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LETTER TO THE EDITOR

Use of double trigonometric series in the solution of plane elasticity problems by calculus of variation, a comment on an error in Timoshenko and Goodier's book *Theory of Elasticity*, 2nd Edition (1951)

1. INTRODUCTION

An error of calculation in the book *Theory of Elasticity* by Timoshenko and Goodier (1951) has been discovered and corrected by us. Besides this, the convergency of the solution by double trigonometric series is also discussed in this article.

2. ERROR OF CALCULATION IN TWO EXAMPLES OF PLATE ANALYSIS†

The calculus of variation in elasticity is a versatile and effective approximate method where double trigonometric series plays an important role. Better convergency can be obtained by using this method for solving various problems such as the bending of rectangular plates simply supported along four edges, the buckling of the same under axial loading, the torsion of rods with rectangular cross-sections etc.

However, attention should be paid to distinguishing the orthogonality of the functions in the interval $[0, a]$ in dealing with this double series. The correct value of the following integration is

$$\int_0^a \sin\frac{m\pi x}{a}\cos\frac{m'\pi x}{a}\,dx = \begin{cases} 0 & \text{if } m \pm m' \text{ is an even number} \\ \dfrac{2am}{\pi(m^2 - m'^2)} & \text{if } m \pm m' \text{ is an odd number} \end{cases} \tag{1}$$

which was mistakenly thought to be

$$\int_0^a \sin\frac{m\pi x}{a}\cos\frac{m'\pi x}{a}\,dx \equiv 0 \tag{2}$$

in two examples of plate analysis by the principle of virtual work in the book by Timoshenko and Goodier (1951).

The first example is a rectangular plate with fixed edges and acted upon by body forces parallel to its plane. The components of displacement have been taken as

$$u = \sum\sum A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$v = \sum\sum B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{3}$$

But, unfortunately, the final expressions for determining the coefficients A_{mn} and B_{mn} were erroneously given in the book by Timoshenko and Goodier (1951) as

† These examples have been deleted in the 1972 edition, without explanation.

$$\frac{Eab\pi^2}{4}\left[\frac{m^2}{a^2(1-v^2)}+\frac{n^2}{2b^2(1+v)}\right]A_{mn}=\int_0^a\int_0^b X\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \quad (4)$$

and

$$\frac{Eab\pi^2}{4}\left[\frac{n^2}{b^2(1-v^2)}+\frac{m^2}{2a^2(1+v)}\right]B_{mn}=\int_0^a\int_0^b Y\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy. \quad (5)$$

The correct results acquired by Xi and Lin (1981) using expressions (1), (3) and Ritz's method are

$$\frac{Eab\pi^2}{4}\left[\frac{m^2}{a^2(1-v^2)}+\frac{n^2}{2b^2(1+v)}\right]A_{mn}-\frac{2Emn}{(1-v)}\sum_{m'}\sum_{n'}B_{m'n'}\frac{m'n'}{(m^2-m'^2)(n^2-n'^2)}$$
$$=\int_0^a\int_0^b X\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \quad (6)$$

$$\frac{Eab\pi^2}{4}\left[\frac{m^2}{2a^2(1+v)}+\frac{n^2}{b^2(1-v^2)}\right]B_{mn}-\frac{2Emn}{(1-v)}\sum_{m'}\sum_{n'}A_{m'n'}\frac{m'n'}{(m^2-m'^2)(n^2-n'^2)}$$
$$=\int_0^a\int_0^b Y\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy. \quad (7)$$

The difference between eqns (4), (5) and (6), (7) is quite obvious. The former represents two independent sets of equations for A_{mn} and B_{mn}, respectively, whereas the latter is coupled for A_{mn} and B_{mn}. Therefore the results of the calculation are bound to be different.

3. CONVERGENCY OF THE SOLUTION

For the sake of simplicity, only one example given in the book by Timoshenko and Goodier (1951) is analyzed here. Assume that the above-mentioned plate is square and in a vertical position (Fig. 1). Let $X = 0$, $Y = \rho g$, $a = b$ and Poisson's ratio $v = 0$. Substitution of these values into eqns (6) and (7) leads to

$$A_{mn}\left(m^2+\frac{n^2}{2}\right)-\frac{8mn}{\pi^2}\sum_{m'}\sum_{n'}\frac{m'n'}{(m'^2-m^2)(n'^2-n^2)}B_{m'n'}=0 \quad (8)$$

$$B_{mn}\left(n^2+\frac{m^2}{2}\right)-\frac{8mn}{\pi^2}\sum_{m'}\sum_{n'}\frac{m'n'}{(m'^2-m^2)(n'^2-n^2)}A_{m'n'}=\frac{16a^2\rho g}{E\pi^4 mn}. \quad (9)$$

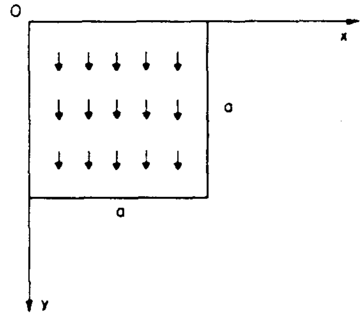

Fig. 1.

Table 1

$(m)_{max}$	$(n)_{max}$	Series term	$u(a/4, a/4)$		$v(a/2, a/2)$	
			This paper	Timoshenko and Goodier (1951)	This paper	Timoshenko and Goodier (1951)
1	1	1	0	0	0.1095	0.1095
2	2	4	0.6671×10^{-2}	0	0.1111	0.1095
3	3	9	0.5417×10^{-2}	0	0.0979	0.0951
5	5	25	0.5308×10^{-2}	0	0.0999	0.0981
9	9	81	0.4647×10^{-2}	0	0.0997	0.0974
11	11	121	0.4757×10^{-2}	0	0.0995	0.0973
12	12	144	0.4757×10^{-2}	0	0.0995	—

Note: the figures for u and v should be multiplied by a factor $a^2\rho g/E$.

In these two equations, $m \pm m'$ and $n \pm n'$ must be odd numbers, and m, n should also be odd numbers in the right-hand term of eqn (9).

These rather tedious simultaneous equations can be solved without much difficulty with the aid of a computer. Only the values of the displacements $u(a/4, a/4)$ and $v(a/2, a/2)$ have been worked out and given in Table 1.

4. CONCLUSIONS

(1) As the book by Timoshenko and Goodier (1951) is a monograph of world-wide fame and has been translated into many different languages, it is worthwhile noting its error of calculation in the two examples of plate analysis so that readers can make due corrections.

(2) By using the formulae provided in this paper, it can be seen from the results listed in Table 1 that, for the example given above, the solution corresponding to $(m)_{max} = (n)_{max} = 11$ can be considered as the precise one and that a result with an error less than 1% can be obtained by setting $(m)_{max} = (n)_{max} = 5$.

(3) It is also shown from Table 1 that it is impossible to obtain the correct horizontal displacement u by using the formulae provided in the book by Timoshenko and Goodier (1951). It is solely in this example (subjected to body forces) that the main displacement v deviates only slightly from the true solution. However, under the action of more complicated loading, the deviation will be larger.

(4) The problem in integration mentioned above may appear in other problems of elasticity. One example is the shear buckling of simply-supported rectangular plates. Therefore this problem deserves attention.

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XING ZHONG XI and JIA JI LIN
Xian Institute of Metallurgy and Construction Engineering
Xian, Shaanxi Province, P.R.C.

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